Comparison of the Kalman Filter with Classical Digital Filter

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In a recent paper Hamilton et al. (1973) evaluated the use of a Kalman filter in a multivariable process industry feedback control system. In addition to analyzing the sensitivity of the Kalman filter to various parameters, they compared its performance to that of an exponential filter commonly used in DDC applications. While poorer performance can be expected with an exponential filter, the very poor, nearly unstable response displayed in their simulation is not necessarily typical of an exponential filter in all applications. In this paper, the outputs of three different types of classical filters are compared under ideal conditions with that of a Kalman filter so that the direct effects on output variables may be observed.

The point at issue is whether the extra design effort and implementation costs of the Kalman filter can be justified by its improved performance. Since the Kalman filter provides a maximum likelihood estimate for systems with suitable statistical qualifications, classical filters must always be somewhat inferior and thus must be justified on some operational basis. Such reasons might range from familiarity with classical filter operating characteristics to the expense of measuring the process inputs used in the Kalman filter.

It is reemphasized here that while classical filters merely provide signal conditioning on observable variables, the Kalman filter yields an estimate for the entire state vector. State vector information may not be useful in some situations, but in others it can provide valuable insight into operations and may constitute indispensable elements of an optimal controller. The availability of this additional information can be an important factor in filter selection. For example, use of the exponential filter in the multivariable control system of Hamilton et al. (1973) necessitated generation of one of the state variables by separate means.

The three types of simple classical filters considered here take the form (Monroe, 1962):

$$c_N = \sum_{1}^{N} a_k \, z_{N-k} \tag{1}$$

For the linear averaging filter, the weighting factors are given by

$$a_k = \frac{1}{N} \tag{2}$$

This filter is equivalent to passing a horizontal line through the most recent section of the data.

For the linear polynomial filter, the weighting factors become

$$a_k = \frac{2(2N-1) - 6k}{N(N+1)} \tag{3}$$

This filter is equivalent to passing a straight, but nonhorizontal, line through a section of the data. In principle, it should allow prediction for a dynamically changing process by extrapolating into the region of future time.

The third type of classical filter tested here is the exponential filter which operates on an infinite set of data points. It forms a moving average that more heavily weights most recent data while the influence of older data points decreases exponentially. Although it can be formulated in the above form, the exponential filter is more easily implemented using the equation

$$c_N = (1-p)c_{N-1} + p\, z_N, \quad 0 \leq p \leq 1 \tag{4}$$

In each of these filters, a trade-off must be made between the degree of smoothing or noise elimination and the responsiveness of the filter to dynamic change in the process. The former can be computed but the latter is more difficult to arrive at by analytical means. Hence in this work the performance of each filter was evaluated by a Monte Carlo simulation approach. A process system was digitally simulated with digitally generated random process inputs and observation noise. These simulations were repeated 100 times with independently generated noise sequences. The average of the square of the difference between the estimate and true value of the observed variable at each point was taken as a criterion of merit. For the Kalman filter, the mean value of this squared difference may be computed as the variance of the filtering error. As will be seen, this theoretical value agrees closely with the simulation results.

The simulated system was a two-stage, plate-type gas absorber similar to that used in other control studies (Koppel, 1968). The second-order system resulting from this model is not a limitation on this study since most chemical processes can be represented as second-order systems within usual limits of measurement error. The results would not be expected to be different if higher order processes were simulated.

In the study by Hamilton et al. (1973) the gain of the Kalman filter was taken as a constant . . . the steady state value. Technically this produces the Wiener filter rather than the more general Kalman filter and, while the former is easier to implement, superior capabilities of the latter under some conditions may influence filter selection. In the present work, the performance of all filters was analyzed through three temporal regions: t_1, the initial region

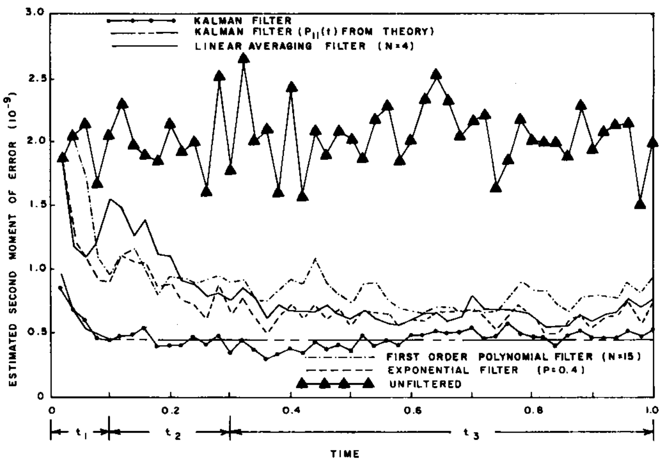

Fig. 1. Squared error of filters averaged over 100 trials; noise level $q/r = 2.5 \times 10^2$.

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Table 1. Percent Noise Transmission of Filters

Noise level	$q/r = 10^3$			$q/r = 2.5 \times 10^2$			$q/r = 0.625 \times 10^2$		
Time period	t_1	t_2	t_3	t_1	t_2	t_3	t_1	t_2	t_3
Kalman filter	44%	43%	42%	32%	22%	22%	26%	11%	10%
Linear averaging									
($N = 2$)	74%	67%	58%	62%	56%	51%	59%	54%	49%
($N = 3$)	95	76	54	59	45	38	51	38	33
($N = 4$)	127	105	61	66	45	33	51	30	27
($N = 5$)	148	154	71	71	53	32	52	28	23
($N = 6$)	148	221	84	71	67	33	52	29	20
($N = 7$)	148	295	99	71	84	35	52	31	19
($N = 8$)	148	369	116	71	101	38	52	34	18
($N = 9$)	148	442	135	71	119	42	52	38	18
($N = 10$)	148	512	157	71	135	47	52	41	19
First-order polynomial									
($N = 5$)	80%	75%	67%	79%	67%	60%	79%	65%	59%
($N = 10$)	80	73	65	79	48	41	79	42	35
($N = 15$)	80	86	83	79	48	39	79	39	27
($N = 20$)	80	86	113	79	48	43	79	39	24
($N = 25$)	80	86	149	79	48	50	79	39	25
Exponential									
($p = 0.1$)	334%	1,079%	465%	133%	282%	119%	84%	84%	33%
($p = 0.2$)	221	393	119	95	107	38	64	37	17
($p = 0.3$)	150	175	64	73	57	29	54	28	20
($p = 0.4$)	107	97	49	62	43	31	51	30	26
($p = 0.5$)	83	68	47	58	43	36	52	36	33
($p = 0.6$)	72	61	50	59	48	44	56	45	42
($p = 0.7$)	70	63	57	64	57	54	63	56	53
($p = 0.8$)	75	71	67	73	69	66	73	68	66
($p = 0.9$)	85	83	82	85	83	81	85	83	81

during which both the process and the filter are in unsteady state; t_2, the dynamic region during which the filter has nearly achieved steady state but the process has not; t_3, the near steady region during which both filter and process changes are slow and amount to little more than dynamic drift. In regions t_2 and t_3 the Kalman filter, being near steady state, is approximately a Wiener filter.

The squared error in the observations at each data point was averaged over the ensemble of 100 simulations. These averages were then time averaged over each of the three dynamic periods described above. Table 1 summarizes this data as percent sum of squares of filtered output as compared to the unfiltered observations. The actual time course of the ensemble average error variation for the middle noise level is plotted in Figure 1. The values for the smoothing parameters N and p for the plot of the classical filter outputs were taken from Table 1 as those having the most desirable overall performance.

The performance of the linear polynomial filter is consistently the least desirable. It is apparently not practical to fit a line to changing noisy data and this filter finds very little use. On the other hand, the Kalman filter in every case during every time period is superior to the linear averaging and exponential filters which are approximately equivalent. This is not surprising since the Kalman filter is a mathematically best estimate. However, the important question may now be considered as to the degree of superiority of the Kalman filter.

During the initial and dynamic time periods, the Kalman filter outperforms the others at all noise levels. The sum of the squared errors is about one-half that of the best exponential or linear averaging filter.

During the steady state period the improvement is not as marked in all situations. At relatively low levels of noise in the observations, ($q/r = 10^3$), the Kalman filter is not a great deal more effective than the exponential filters. At higher noise levels in the observations ($q/r = 0.625 \times 10^2$), the Kalman filter showed a squared residual error that averaged about one-half of that of the best exponential filter. However, at these latter noise conditions the total noise reduction of all of the filters was quite good and for many applications 80% reduction in the squared error may be as effective, in a practical sense, as 90% reduction.

SUMMARY

The Kalman filter can always provide a better estimate than a classical filter. For processes subject to large rapid changes, the unsteady state Kalman filter clearly out performs all other filters; however, at relatively low noise levels in the observations at near steady state conditions, the Kalman filter is only marginally superior to some classical filters. At higher levels of noise in the observations in near steady state conditions, the Kalman filter is clearly superior, but the difference may not be important since all filters perform quite well. The Kalman filter provides more than just the filtered output for one variable and these other results may be of great importance for some situations.

In general, the Kalman filter, which has been described as a least squares estimator reflected through system dynamics, is superior for those applications where known system dynamics influence the output variable in an important way. In such situations, its substitution for classical filters is likely to be most easily justified.

Thus in the example of Hamilton et al. (1973) even the best exponential filter, which could not compensate for system dynamics, led to oscillations in the tight optimal feedback control loop. As pointed out however in the original work (Hamilton, 1972), loosening of the loop with a lag designed controller might improve control significantly although of course not to the optimal level.

NOTATION

a_k = kth weighting factors in filter equation
c_N = Nth filter output
p = exponential filter weighting factor
q = process noise covariance
r = measurement noise covariance
t_i = ith dynamic time period
z_N = nth observation

LITERATURE CITED

Hamilton, J. C., "An Experimental Investigation of State Estimation in Multivariable Control Systems," M. Sc. thesis, Univ. Alberta, Edmonton, Canada (1972).

Hamilton, J. C., D. E. Seborg, and D. G. Fisher, "An Experimental Evaluation of Kalman Filtering," *AIChE J.*, **19**, 901 (1973).

Koppel, L. B., *Introduction to Control Theory*, Prentice Hall, Englewood Cliffs, N. J. (1968).

Kneile, R. G., "Comparison of Kalman Filter with Classical Filters," M.S. thesis, University of Missouri, Columbia (1973).

Monroe, A. J., *Digital Processes for Sample Data Systems*, Wiley, New York (1962).

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Mechanical Equilibrium for Eccentric Rotating Disks

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Dynamic mechanical measurements can be very helpful in characterizing the molecular structure of polymer melts and solutions. Generally dynamic properties have been determined from a sinusoidally varying deformation (Ferry, 1970). To evaluate the dynamic material properties it is necessary to determine the in phase and quadrature components of the stress in reference to the sinusoidal deformation. Measuring these components can be difficult.

Gent (1960) has suggested an alternate method for dynamic shear measurements, the flow between two parallel disks rotating about noncoincident axes. This eccentric rotating disk (ERD) geometry, shown in Figure 1, gives forces in the x and y directions which are steady in time and yield the dynamic quantities η' and G' directly. Recently we have shown that these correlate very well with results of sinusoidal tests (Macosko and Davis, 1972).

Previous theoretical analyses have assumed that both the upper and lower disk rotate with the same angular velocity (Blyler and Kurtz, 1967; Bird and Harris, 1968; and others reviewed by Macosko and Davis, 1972). However, in all experimental devices reported, one disk is driven and the other follows through viscous drag. This note provides an analysis of the actual experimental situation and tests this analysis experimentally.

If the lower disk rotates at the same angular velocity as the upper, driven disk ($\Omega_1 = \Omega_2$ in Figure 1), then the stresses produced by the flow will cause forces F_{x1} and F_{x2} to act on the axes of lower and upper disk, respectively. Thus a net torque, aF_{x2} will act about the axis of the lower disk. If the lower disk is free to rotate it will accelerate unless this torque is balanced. To balance this torque we suggest that the lower disk rotates at a slightly smaller angular velocity, causing a small torsional flow to occur, generating an opposite torque. Additional torsional flow can occur due to any drag in the lower bearing.

NEWTONIAN FLUID

We can solve the Navier Stokes equations for the flow between ERD with the boundary conditions of solid body rotation at each disk surface:

$$\begin{aligned} &\text{at} \quad z = 0 \quad v_x = -\Omega_1 y \qquad v_y = \Omega_1 x \quad v_z = 0 \\ &\text{at} \quad z = h \quad v_x = -\Omega_2 y + \Omega_2 a \quad v_y = \Omega_2 x \quad v_z = 0 \end{aligned} \tag{1}$$

The solution is simplest in cartesian coordinates and generally follows that of Abbott and Walters (1970) and Goldstein (1971). For a fluid with zero density we find the following kinematics:

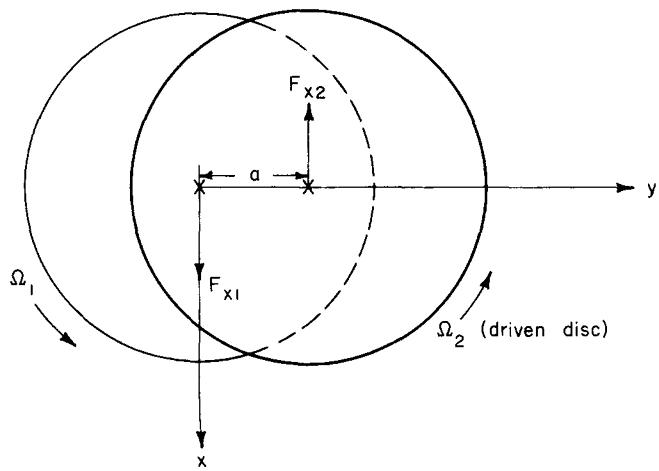

Fig. 1. Schematic diagram of ERD (top view), origin of cartesian coordinates is on lower disc.